

ORDINARY REDUCTION OF K3 SURFACES

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Let K be a number field and A an abelian variety of positive dimension over K . It is well known that A has good reduction at all but finitely many (non-archimedean) places of K . It is natural to ask whether among those reductions there is ordinary one. In the most optimistic form the precise question sounds as follows.

Is it true that there exists a finite algebraic field extension L/K and a density 1 set S of places of L such that $A \times L$ has ordinary good reduction at every place from S ?

The positive answer is known for elliptic curves (Serre [21]), abelian surfaces (Ogus [19]) and certain abelian fourfolds and threefolds [15, 16, 25].

One may ask a similar question for other classes of (smooth projective) algebraic varieties. The aim of this note is to settle this question for K3 surfaces. Recall that an (absolutely) irreducible smooth projective surface X over an algebraically closed field is called a K3 surface if the canonical sheaf Ω_X^2 is isomorphic to the structure sheaf \mathcal{O}_X and $H^1(X, \mathcal{O}_X) = \{0\}$.

Our main result is the following statement.

Theorem 0.1. *Let X be a K3 surface that is defined over a number field K . Then there exists a finite algebraic field extension L/K and a density 1 set $\Sigma(L, X)$ of (non-archimedean) places of L such that $X \times_K L$ has ordinary good reduction at every place $v \in \Sigma(L, X)$.*

Remark 0.2. The case of Kummer surfaces follows from the result of Ogus concerning the existence of ordinary reductions of abelian surfaces. When the *endomorphism field* E of $X \times_K \mathbf{C}$ [28, Th. 1.6] is totally real (e.g., the Picard number is odd), the assertion of Theorem 0.1 was proven by Tankeev [25].

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After this paper had appeared on arXiv and was submitted, we received a letter from professor K. Joshi who brought to our attention his joint preprint with C.Rajan, <http://arxiv.org/abs/math/0110070>. Its Section 6 contained the proof of the existence of a positive density set of places with ordinary reduction for any K3-surface over a number field. Some of their arguments (but not all) are similar to the ones we used. Unfortunately, this result and its generalizations did not appear in the printed version [31].

1. K3 SURFACES OVER FINITE FIELDS

Let k be a finite field of characteristic p , let \bar{k} be its algebraic closure, let Y be a K3 surface defined over k and $\bar{Y} = Y \times \bar{k}$.

1.1. Let ℓ be a prime different from p and

$$P_2(Y, t) = 1 + \sum_{i=1}^{22} a_i t^i$$

the characteristic polynomial of the Frobenius endomorphism Fr of \bar{Y} in the second ℓ -adic cohomology group $H^2(\bar{Y}, \mathbf{Z}_\ell)$. It is known (P. Deligne [6]; Piatetskii–Shapiro and I.R. Shafarevich [20]) that $P_2(Y, t)$ lies in $1 + t\mathbf{Z}[t]$ and does not depend on the choice of ℓ ; in addition, all reciprocal roots of $P_2(Y, t)$ have (archimedean) absolute value $q = \#(k)$. It is also known [7, Cor. 1.10 on p. 63] that Fr acts on the \mathbf{Q}_ℓ -vector space

$$H^2(\bar{Y}, \mathbf{Q}_\ell) = H^2(\bar{Y}, \mathbf{Z}_\ell) \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell$$

as a semisimple (i.e., diagonalizable over an algebraic closure of \mathbf{Q}_ℓ) linear operator.

Let us split $P_2(Y, t)$ into a product of linear factors (over $\bar{\mathbf{Q}}$)

$$P_2(Y, t) = \prod_{i=1}^{22} (1 - \alpha_i t)$$

where α_i are the reciprocal roots of $P_2(Y, t)$. Clearly,

$$a_1 = - \sum_{i=1}^{22} \alpha_i.$$

Let $L = \mathbf{Q}(\alpha_1, \dots, \alpha_{22})$ be the splitting field of $P_2(Y, t)$: it is a finite Galois extension of \mathbf{Q} . We write \mathcal{O}_L for the ring of integers in L . Clearly, all $\alpha_i \in \mathcal{O}_L$. For each field embedding $L \hookrightarrow \mathbf{C}$ the image of every α_i has absolute (archimedean) value q . This implies that if α is one of the reciprocal roots then its *complex-conjugate* $\bar{\alpha} = q^2/\alpha$ is also one of the reciprocal roots; in particular, q^2/α also lies in \mathcal{O}_L . It follows that if \mathfrak{B} is a maximal ideal in \mathcal{O}_L that does *not* lie above p then α is a \mathfrak{B} -adic unit.

1.2. In order to describe the p -adic behavior of the reciprocal roots, one has to use their crystalline interpretation and use a variant of Katz's conjecture proven in [3, Sect. 8]. Recall [7, Prop. 1.1 on p. 59] that

$$h^{0,2}(\bar{Y}) = 1, h^{1,1}(\bar{Y}) = 20$$

and the crystalline cohomology groups of \bar{Y} have no torsion. Combining Theorem 8.39 on p. 8-47 of [3] and Example 2 on p. 659 of [13], one concludes ([2, Examples on pp. 90–91], [1]) that there exists a certain invariant $h = h(Y)$ of a K3 surface Y called its *height* that enjoys the following properties.

The height h is either a positive integer ≤ 10 or ∞ . A K3 surface is called ordinary if $h = 1$ and supersingular if $h = \infty$. Let \mathfrak{P} be (any) maximal ideal in \mathcal{O}_L that lies above p and let

$$\text{ord}_{\mathfrak{P}} : L^* \rightarrow \mathbf{Q}$$

be the discrete valuation map attached to \mathfrak{P} and normalized by condition

$$\text{ord}_{\mathfrak{P}}(q) = 1.$$

If $h = \infty$ then every $\text{ord}_{\mathfrak{P}}(\alpha) = 1$. If $h \leq 10$ then the sequence

$$\text{ord}_{\mathfrak{P}}(\alpha_1), \dots, \text{ord}_{\mathfrak{P}}(\alpha_{22})$$

consists of rational numbers $(h-1)/h, 1, (h+1)/h$: both numbers $(h-1)/h$ and $(h+1)/h$ occur h times in the sequence while the number 1 occurs $(22-2h)$ times.

- Remark 1.3.** (i) Suppose that $h = \infty$. If α is one of the reciprocal roots then α/q is a \mathfrak{P} -adic unit for every \mathfrak{P} dividing p . It follows that α/q is unit in \mathcal{O}_L . On the other hand, since all archimedean absolute values of α are equal to q , we conclude that all archimedean absolute values of α/q are equal to 1. By Kronecker's theorem, α/q is a root of unity.
- (ii) If $h \neq 1$ then all $\text{ord}_{\mathfrak{P}}(\alpha_i)$ are positive numbers and therefore all the reciprocal roots lie in \mathfrak{P} .
- (iii) If $h = 1$ then $(h-1)/h = 0$ and therefore there is exactly one reciprocal root α that does not lie in \mathfrak{P} and this root is simple.

Lemma 1.4. *Y is ordinary if and only if a_1 is not divisible by p .*

Proof. Since $a_1 = -\sum_{i=1}^{22} \alpha_i$, It follows from Remark 1.3 that a_1 does not lie in \mathfrak{P} if and only if $h = 1$. Now one has only to recall that $a_1 \in \mathbf{Z}$ and $\mathbf{Z} \cap \mathfrak{P} = p\mathbf{Z}$. \square

Additional information about K3 surfaces over finite fields could be found in [17, 18, 30, 27].

2. A TECHNICAL RESULT

Let K be a number field and \mathcal{O}_K its ring of integers. Let \bar{K} be an algebraic closure of K . Let $\text{Gal}(K) = \text{Gal}(\bar{K}/K)$ be the absolute Galois group of K . Let X be a K3 surface that is defined over a number field K and let \bar{X} be the K3 surface $X \times_K \bar{K}$ over \bar{K} .

Remark 2.1. There exists a nowhere vanishing regular exterior 2-form on \bar{X} that is defined over K . Indeed, pick any regular nowhere vanishing 2-form $\bar{\omega}$ on \bar{X} . Then for each $\sigma \in \text{Gal}(K)$ the 2-form $\sigma\bar{\omega}$ coincides with $c_\sigma \cdot \bar{\omega}$ for a certain non-zero $c_\sigma \in \bar{K}^*$. We get a Galois cocycle $\sigma \mapsto c_\sigma$. By Hilbert's Theorem 90, there exists $a \in \bar{K}^*$ such that

$$c_\sigma = \sigma(a)/a \quad \forall \sigma \in \text{Gal}(K).$$

It follows that the 2-form $\omega = a^{-1}\bar{\omega}$ is Galois-invariant. As a corollary, we obtain that the canonical (invertible) sheaf $\Omega_{X/K}^2$ is isomorphic to the structure sheaf \mathcal{O}_X .

If S is a finite set of primes then let us consider the localization $\Lambda = \Lambda_S := \mathcal{O}_K[S^{-1}]$ of \mathcal{O}_K with respect to S . Clearly, Λ is a Dedekind ring,

$$\mathcal{O}_K \subset \Lambda \subset K$$

and $\text{Spec}(\mathcal{O}_K) \setminus \text{Spec}(\Lambda)$ is a finite set of maximal ideals in \mathcal{O}_K , whose residual characteristic lies in S .

2.2. There exists a finite set of primes S and a smooth projective morphism $\mathcal{X} \rightarrow \text{Spec}(\Lambda)$ that enjoy the following properties:

- The generic fiber \mathcal{X}_K coincides with X .
- The invertible (canonical) sheaf $\Omega_{\mathcal{X}/\Lambda}^2$ is isomorphic to the structure sheaf $\mathcal{O}_{\mathcal{X}}$.

- The cohomology group $H^1(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is a free Λ -module of finite rank.
- For every commutative Λ -algebra B the canonical map

$$H^1(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \otimes_{\Lambda} B \rightarrow H^1(\mathcal{X}_B, \mathcal{O}_{\mathcal{X}_B})$$

is an isomorphism. Here

$$\mathcal{X}_B = \mathcal{X} \times_{\text{Spec}(\Lambda)} \text{Spec}(B).$$

Since $H^1(\bar{X}, \mathcal{O}_{\bar{X}}) = \{0\}$, we conclude that $H^1(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) = \{0\}$ and therefore $H^1(X_s, \mathcal{O}_{X_s}) = \{0\}$ for any geometric point s of $\text{Spec}(\Lambda)$. In particular, \mathcal{X}_s is a K3 surface.

The assertion follows from general results about the existence of smooth projective models [9, pp. 157–158, Prop. A.9.1.6] and base change theorems for Hodge cohomology [10, Sect. 8, pp. 203–205] (see also [14, Sect. 5]).

We call such schemes $\mathcal{X} \rightarrow \text{Spec}(\Lambda)$ *good models* of X .

3. ORDINARY REDUCTIONS

Let X be a K3 surface over a number field K . Let us pick a prime $\ell > 2 \cdot 22$ and consider the corresponding 22-dimensional ℓ -adic representation [21, 22]

$$\rho_{2,X} : \text{Gal}(K) \rightarrow \text{Aut}_{\mathbf{Z}_{\ell}}(H^2(\bar{X}, \mathbf{Z}_{\ell})) \subset \text{Aut}_{\mathbf{Q}_{\ell}}(H^2(\bar{X}, \mathbf{Q}_{\ell})).$$

We write $G_{\ell,X,K}$ for the image $\rho_{2,X}(\text{Gal}(K))$: it is a closed compact subgroup of $\text{Aut}_{\mathbf{Z}_{\ell}}(H^2(\bar{X}, \mathbf{Z}_{\ell}))$; in particular, it is an ℓ -adic Lie subgroup of $\text{Aut}_{\mathbf{Q}_{\ell}}(H^2(\bar{X}, \mathbf{Q}_{\ell}))$. One may view $G_{\ell,X,K}$ as the Galois group of the infinite Galois extension $\bar{K}^{\ker(\rho_{2,X})}/K$ where $\bar{K}^{\ker(\rho_{2,X})}$ is the subfield of $\ker(\rho_{2,X})$ -invariants in \bar{K} .

We write Id for the identity automorphism of $H^2(\bar{X}, \mathbf{Z}_{\ell})$. Then the set

$$\mathfrak{Z} := \{c \cdot \text{Id} \mid c \in \mathbf{Z}_{\ell}^*\} \bigcap G_{\ell,X,K}$$

is a closed normal ℓ -adic Lie subgroup in $G_{\ell,X}$.

Lemma 3.1. *The subgroup \mathfrak{Z} is not open in $G_{\ell,X,K}$. In particular, $\dim(\mathfrak{Z}) < \dim(G_{\ell,X})$.*

Proof. Suppose that \mathfrak{Z} is open in $G_{\ell,X}$. Then it has finite index and there exists a finite Galois extension K'/K such that

$$G_{\ell,X,K'} = \rho_{2,X}(\text{Gal}(K')) = \mathfrak{Z},$$

i.e., $\text{Gal}(K')$ acts on $H^2(\bar{X}, \mathbf{Q}_{\ell})$ via scalars. It follows that $\text{Gal}(K')$ acts on the twisted ℓ -adic cohomology group $H^2(\bar{X}, \mathbf{Q}_{\ell})(1)$ also via scalars. This implies that $H^2(\bar{X}, \mathbf{Q}_{\ell})(1)^{\text{Gal}(K')}$ is either $H^2(\bar{X}, \mathbf{Q}_{\ell})(1)$ or zero. However, it is known [26, Th. 5.6 on p. 80] that

$$H^2(\bar{X}, \mathbf{Q}_{\ell})(1)^{\text{Gal}(K')} = \text{NS}(\bar{X})^{\text{Gal}(K')} \otimes \mathbf{Q}_{\ell}.$$

(This assertion is the Tate conjecture for K3 surfaces that follows from the corresponding results of Faltings concerning abelian varieties [8].)

Since the Néron–Severi group $\text{NS}(\bar{X})$ of \bar{X} is a (non-zero) free commutative group of rank $\leq 20 < 22$, we conclude that

$$H^2(\bar{X}, \mathbf{Q}_{\ell})(1)^{\text{Gal}(K')} \neq H^2(\bar{X}, \mathbf{Q}_{\ell})(1)$$

and therefore

$$H^2(\bar{X}, \mathbf{Q}_{\ell})(1)^{\text{Gal}(K')} = \{0\}.$$

However, this is not the case, because there is a hyperplane section of X that is defined over K' (and even over K) and its ℓ -adic cohomology class is Galois-invariant and not zero. The obtained contradiction proves the Lemma. \square

Remarks 3.2. Combining results of [6] and [4, 5], one may prove that $\dim(\mathfrak{z}) = 1$. We refer to [24] for other applications of the Tate conjecture [26] and its variants to arithmetic of K3 surfaces over number fields.

The following statement and its proof are inspired by results of N. Katz and A. Ogus [19, Prop. 2.7.2 on p. 371].

Lemma 3.3. *Suppose that a prime p and an element $u \in G_{\ell, X, K}$ enjoy the following properties:*

- (i) $p - 1$ is divisible by ℓ .
- (ii) $u \in \text{Id} + \ell \cdot \text{End}_{\mathbf{Z}_\ell}(H^2(\bar{X}, \mathbf{Z}_\ell))$.
- (iii) The characteristic polynomial

$$P_u(t) = \det(1 - tu, H^2(\bar{X}, \mathbf{Q}_\ell)) = 1 + b_1 t + \cdots + b_{22} t^{22}$$

lies in $\mathbf{Z}[t]$.

- (iv) *Let us split $P_u(t)$ into a product of linear factors*

$$P_u(t) = \prod_{i=1}^{22} (1 - \beta_i t).$$

Then all the reciprocal roots $\beta_1, \dots, \beta_{22}$ of $P_u(t)$ have the same archimedean absolute value p .

- (v) b_1 is divisible by p .

Then $p^{-1}u$ is a unipotent linear operator in $H^2(\bar{X}, \mathbf{Q}_\ell)$. In particular, if u is semisimple then $u = p \cdot \text{Id}$.

Proof. So, $-b_1 = pc$ for some integer c . Notice that $\beta_1, \dots, \beta_{22}$ are the eigenvalues of u and $-b_1 = \sum_{i=1}^{22} \beta_i$ is the trace of u .

The congruence condition for u implies that all $(\beta_i - 1)/\ell$ are algebraic integers and therefore the integer $-b_1 = pc$ is congruent to 22 modulo ℓ . Since $p - 1$ is divisible by ℓ , it follows that c is congruent to 22 modulo ℓ .

It is also clear that the absolute value of $(-\beta_1)$ does not exceed $22 \cdot p$ and therefore $|c| \leq 22$. Taking into account that $\ell > 2 \cdot 22$ and c is congruent to 22 modulo ℓ , we conclude that $c = 22$, i.e., $-b_1 = 22 \cdot p$. Since $-b_1$ is a sum of 22 complex numbers $\beta_1, \dots, \beta_{22}$ of absolute value p , it follows that all $\beta_i = p$, i.e., all eigenvalues of u are equal to p , which means that $p^{-1}u$ is unipotent. \square

3.4. Choose a good model $\mathcal{X} \rightarrow \text{Spec}(\Lambda)$ of X (as in Sect. 2). Let $\mathfrak{v} \in \text{Spec}(\Lambda)$ be a closed point, whose residual characteristic $p = p(\mathfrak{v})$ is different from ℓ . Then $\rho_{2, X}$ is unramified at \mathfrak{v} and one may associate to \mathfrak{v} a Frobenius element $\text{Fr}_{\mathfrak{v}} \in G_{\ell, X, K}$ ([21, 22], [29, Sect. 4]) that is defined up to conjugacy. Let us consider the corresponding closed fiber $\mathcal{X}(\mathfrak{v})$, which is a K3 surface over the (finite) residue field $k(\mathfrak{v})$. Let $\overline{k(\mathfrak{v})}$ be an algebraic closure of $k(\mathfrak{v})$ and $\overline{\mathcal{X}(\mathfrak{v})} = \mathcal{X}(\mathfrak{v}) \times_{k(\mathfrak{v})} \overline{k(\mathfrak{v})}$. The Frobenius endomorphism $\text{Fr} : \overline{\mathcal{X}(\mathfrak{v})} \rightarrow \overline{\mathcal{X}(\mathfrak{v})}$ acts on $H^2(\overline{\mathcal{X}(\mathfrak{v})}, \mathbf{Q}_\ell)$ and there exists an isomorphism of \mathbf{Q}_ℓ -vector spaces

$$H^2(\overline{\mathcal{X}(\mathfrak{v})}, \mathbf{Q}_\ell) \cong H^2(\bar{X}, \mathbf{Q}_\ell)$$

such that the action of Fr becomes the action of Fr_v^{-1} ([21, 22], [29, Sect. 4]); in particular, we have the coincidence of the corresponding characteristic polynomials, i.e.,

$$P_2(X(v), t) = \det(1 - t\text{Fr}, H^2(\overline{X(v)}, \mathbf{Q}_\ell)) = \det(1 - t\text{Fr}_v^{-1}, H^2(\bar{X}, \mathbf{Q}_\ell)).$$

In particular, the reciprocal roots of $P_2(X(v), t)$ are exactly the eigenvalues of Fr_v^{-1} ; in addition, it follows from the semisimplicity of the Frobenius endomorphism (Subsect. 1.1) that Fr_v^{-1} and (therefore) Fr_v are semisimple linear operators in $H^2(\bar{X}, \mathbf{Q}_\ell)$. So, if

$$P_2(X(v), t) = 1 + \sum_{i=1}^{22} a_i(v)t^i \in \mathbf{Z}[t]$$

then the integer $-a_1(v)$ coincides with the trace of Fr_v^{-1} in $H^2(\bar{X}, \mathbf{Q}_\ell)$. It follows from Lemma 1.4 that $X(v)$ is ordinary if and only if $a_1(v)$ is *not* divisible by $p(v)$.

3.5. Suppose that

$$G_{\ell, X, K} \subset \text{Id} + \ell \text{End}_{\mathbf{Z}_\ell}(H^2(\bar{X}, \mathbf{Z}_\ell)),$$

K contains a primitive ℓ th root of unity. Suppose also that the residue field $k(v)$ is a prime (finite) field \mathbf{F}_p of characteristic $p = p(v)$. Then $p - 1$ is divisible by ℓ .

Let us assume that $a_1(v)$ is divisible by p . Using the results of Subsect. 1.1 and 3.4, we may apply Lemma 3.3 to $u = \text{Fr}_v^{-1}$ and conclude that $\text{Fr}_v^{-1} = p \cdot \text{Id}$, i.e., $\text{Fr}_v = p^{-1} \cdot \text{Id} \in \mathfrak{Z}$.

This proves that if (instead) we assume that Fr_v does *not* belong to \mathfrak{Z} then $a_1(v)$ is *not* divisible by p and, thanks to the last assertion of Subsect. 3.4, $X(v)$ is ordinary.

It is well-known [12, Theorems 1.112 and 1.113 on p. 83] that the set of v 's with prime residue fields has density one. On the other hand, a result of Serre [23, Sect. 4.1, Cor. 1] applied to $G = G_{\ell, X, K}$ and $C = \mathfrak{Z}$ and combined with Lemma 3.1 implies that the set of v 's with $\text{Fr}_v \in \mathfrak{Z}$ has density zero. It follows that (under our assumptions on K) the set of v 's with ordinary reduction $X(v)$ has density one.

Proof of Theorem 0.1. So, K is an arbitrary number field. There exists a finite Galois extension L/K such that L contains a primitive ℓ th root of unity and

$$G_{\ell, X, L} \subset \text{Id} + \ell \cdot \text{End}_{\mathbf{Z}_\ell}(H^2(\bar{X}, \mathbf{Z}_\ell)).$$

Now the result follows from the last assertion of Subsect. 3.5 applied to L (instead of K .)

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